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Classical Geometry of Bosonic String Dynamics

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We develop a treatment of bosonic strings on a general curved background in which the volume element and the coordinates of the worldsheet are related in a similar way as canonically conjugate quantities in mechanics. The resultant formalism is a particular variant of the multi-phase-space approach to classical field theory put forward by Kijowski, Tulczyjew, and others. We study conservation laws within this framework and find that all conserved quantities are related to point symmetries, i.e., isometries of the underlying spacetime. Thus, the symmetries of relativistic mechanics coming from Killing tensors have no analogue here. We furthermore deduce from the present scheme the covariant version of the usual phase space.

1. INTRODUCTION

The concept of a string is a direct generalization of the relativistic point particle. While this is apparent already in the standard formulation of the theory (Scherk, 1975), there is a sense in which this formulation is not very natural if judged by the standards of relativistic mechanics. To explain this point, let us recall that a theory of classical relativistic particles has two ingredients: One is the idea of a particle as a certain timelike worldline in spacetime. Let us call this the spacetime viewpoint. The other input comes from the description of the particle by an action principle and consists in viewing the particle as tracing out a trajectory in a space Γ which is equipped with a symplectic structure. Let us call this the phase space viewpoint. Luckily, both structures are closely related, namely Γ is given by T^*M, the cotangent bundle of spacetime with its natural symplectic structure. Furthermore, the path in Γ is a "lift" of the worldline in M.

Now consider strings. From the spacetime viewpoint a string sweeps out a timelike extremal worldsheet in spacetime. When one now tries to fit strings into the phase space picture, there seems to be only one way: this is by looking at the "space of all strings in spacetime." Since strings are

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extended objects, their motion is governed by partial (as opposed to ordinary) differential equations. Thus, due to the nature of these equations, phase space becomes an infinite-dimensional space, which, in fact, carries a (pre-) symplectic form. This form, i.e., its value at a point, is now given by a certain integral over cross sections of the worldsheet corresponding to that point. Hence, this phase space, from the spacetime viewpoint, is a global thing in which the picture of the string as a local object in spacetime seemingly disappears. In addition, this space has the disadvantage of not possessing a natural cotangent bundle structure, since introducing coordinates and canonically conjugate momenta involves a decomposition of strings into space and time.

The above description of strings (which, with suitable qualifications, could be given for any classical field theory derived from an action principle—relativistic or nonrelativistic) suggests the question of whether there does not exist a finite-dimensional space which can play the role of the cotangent bundle $\Gamma = T^*M$ in the case of particles. We are thus led to the geometrical approach to the calculus of variations of multiple integrals, which, since the end of the 19th century, has been studied mainly by pure mathematicians (Volterra, 1890; Caratheodory, 1929; Dedecker, 1953) and has somewhat gained in attention recently (Kijowski and Tulczyjew, 1979; Kastrup, 1983; Binz *et al.*, 1986; Gotay *et al.*, 1988). Unfortunately, there is in general no unique or clear-cut finite-dimensional generalization of symplectic geometry to multidimensional action principles (Dedecker, 1977). So, in order to find the "right" answer, it is perhaps best to rely on a case-by-case analysis.

At this stage we invoke a second important feature of relativistic mechanics we have so far ignored, but which one would like to copy in string theory. This concerns the reparametrization invariance of the theory and the way in which it is enforced: Namely, this is done by constraining the particle momentum p_a to lie on the mass shell:

$$p_a p_b g^{ab} = -m^2 \qquad (m > 0) \tag{1.1}$$

where m is the mass of the particle. Here p_a is not to be viewed as a velocity, but as a momentum related to velocity by $p_a = m(-\dot{x}^2)^{-1/2} g_{ab} \dot{x}^b$. In particular, equation (1.1) has nothing to do with fixing the worldline parameter to be proper time, as is sometimes believed—just as the analogous constraint for strings involving the string tension has nothing to do with partially fixing the parametrization by going to the conformal gauge. Rather, this constraint should be viewed as saying that $(1/m)p_a$ is the same as the *volume element* on the worldline induced by the metric of spacetime. In a good, that is to say, manifestly reparametrization-invariant theory of strings it thus seems desirable to deemphasize parametrization-dependent objects such as the differential of the map embedding the string into spacetime—which figures prominently in the standard formulation as a velocity-type variable—and put to the forefront p_{ab}, the induced volume element on the worldsheet, which just depends on orientation.

We sum up our discussion of relativistic mechanics by saying that its structure is very special in two respects: First, there is a "preestablished harmony" between the spacetime and the phase space picture expressed by the equation $\Gamma = T^*M$. Second, there is a relation between velocities in M and momenta in Γ which is reparametrization-invariant by virtue of the mass shell constraint $p_a p_b g^{ab} = -m^2$.

In Section 2 it is our aim to write down a formulation of classical strings which mimicks the above two features of relativistic mechanics [for the mechanics case see, e.g., Sniatycki and Tulczyjew (1971)]. The space T^*M of mechanics now gets replaced by a "multisymplectic" space Γ, namely $\Gamma = \Lambda^2 T^*M$, the bundle of antisymmetric covariant 2-tensors $p_{ab} = p_{[ab]}$ over M. The space Γ carries a basic 2-form and a "canonical" 3-form Ω, in contrast to the basic 1-form and the symplectic form of mechanics. The dynamical equations are derived from a manifestly reparametrization-invariant action principle in the constraint submanifold $\mathscr{P}$ obtained from Γ by the string tension condition $p_{ab}p_{cd}g^{ac}g^{bd} = -2T^2$. In Section 3 we study conservation laws. We find, as opposed to the mechanics case, that the only symmetries of the theory are the ones coming from a symmetry of spacetime. For the latter we write down the corresponding conserved quantities. In Section 4 we apply the method of Kijowski and Szczyrba (1976) to obtain the covariant version of the infinite-dimensional space of all strings, referred to at the beginning of this section. Such covariant phase spaces have recently been studied by a number of authors (Woodhouse, 1980; Ashtekar *et al.*, 1987; Crnković, 1988). Section 5 gives a summary and discussion of our results. In the Appendix we outline the local differential geometry underlying the minimal surface equation when formulated in terms of the quantity of prime interest in this paper, namely the volume element of the surface.

Let us remark, finally, that ideas loosely related to ours have been pursued in Nambu (1980) and Kastrup and Rinke (1981), from the point of view of a Hamilton–Jacobi type theory of strings.

2. THE BOSONIC STRING IN MULTISYMPLECTIC FORM

The (M, g_{ab}) be a spacetime of arbitrary dimension $n > 2$, the signature of g_{ab} being $(- + \cdots +)$. Define the extended multisymplectic phase space Γ by $\Gamma = \Lambda^2 T^*M$. In canonical coordinates (p_{ab}, x^c) we can write down a 2-form Θ and a 3-form Ω as

$$\Theta = p_{ab}\, dx^a \wedge dx^b \tag{2.1}$$

$$\Omega = d\Theta = dp_{ab} \wedge dx^a \wedge dx^b \tag{2.2}$$

(For reasons of convenience we sum over all a and b, whereas, of course, the p_{ab} are independent coordinates only for $a<b$. It will be clear that properly taking this into account would just introduce awkward factors of 1/2 which drop out of final results.) Obviously Θ, and hence Ω, is naturally defined. Next take as constraint submanifold the subbundle $\mathcal{P}_T = \mathcal{P}$ of Γ given by $H(p, x) = 0$, where

$$H(p, x) = \tfrac{1}{2} p_{ab} p_{cd} g^{ac}(x) g^{bd}(x) + T^2 \qquad (T>0) \tag{2.3}$$

$\mathcal{P}$ carries a 2-form $\vartheta = \Theta_{|\mathcal{P}}$ and a 3-form $\omega = d\vartheta$.

Let Q be either a square (for open strings) or $S^1 \times \mathbf{R}$ (for closed strings). We consider embeddings $s: Q \to M$, given locally by $y^\alpha \mapsto x^a = X^a(y^\alpha)$, such that the embedded surface S is timelike. Given an orientation for Q, we can lift s into an embedding $\sigma: Q \to \mathcal{P}$ by setting $y^\alpha \mapsto (p_{ab} = P_{ab}(y^\alpha)$, $x^c = X^c(y^\alpha))$, where (for the notation see the Appendix)

$$P_{ab} = 2T[(\dot{X}X')^2 - \dot{X}^2 X'^2]^{-1/2} g_{ac}(X) g_{bd}(X) \dot{X}^{[c} X'^{d]} \tag{2.4}$$

Let now σ be an embedding of Q into $\mathcal{P}$. We define the action $A(\alpha)$ by

$$A(\sigma) = \int_Q \sigma^* \vartheta \tag{2.5}$$

Note that $A(\sigma)$ does not depend on the parametrization of Q. When σ is a lift of an embedding s in the above sense, which it need not be *a priori*, $A(\sigma) = \mathcal{A}(X; \dot{X}, X')$ is, of course, T times the induced surface area of S, that is to say, the Nambu action for s.

In order to find the critical maps σ of the functional A, take a family σ_ε of maps and define the vector field Y along $\sigma = \sigma_0$ by $Y = (d/d\varepsilon)|_{\varepsilon=0} \sigma_\varepsilon$. From standard formulas in differential geometry (see, e.g., Michor, 1980) we find

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} A(\sigma_\varepsilon) = \int_Q \sigma^* \mathcal{L}_Y \vartheta = \int_Q \sigma^* [Y \lrcorner\, d\vartheta + d(Y \lrcorner\, \vartheta)] \tag{2.6}$$

Supposing σ_ε to be independent of ε on the boundary ∂Q of Q, so that Y vanishes on ∂Q but is otherwise arbitrary, we obtain as condition of criticality for σ the relation

$$\sigma^*(Y \lrcorner\, \omega) = 0 \qquad \text{for all } Y \text{ tangent to } \mathcal{P} \tag{2.7}$$

Equation (2.7) is the usual form of equations of motion in the multisymplectic scheme (see, e.g., Kijowski, 1973). In order to evaluate (2.7), note that, with i being the inclusion of $\mathcal{P}$ into Γ, equation (2.7) can be written as

$$\sigma^* i^*(Y \lrcorner\, \Omega) = (i \circ \sigma)^*(Y \lrcorner\, \Omega) = 0 \tag{2.8}$$

for all vector fields on Γ which, at points of $\mathcal{P}$, are tangential to $\mathcal{P}$. Locally $i \circ \sigma$ is given by $(p_{ab}, x^c) = (P_{ab}(y), X^c(y))$ such that $H(P, X) = 0$, and Y is given by $Y = \alpha_{ab}(p, x)\, \partial/\partial p_{ab} + \beta^a(p, x)\, \partial/\partial x^a$ subject to

$$\alpha_{ab}(P, X) g^{ac}(P, X) g^{bd}(P, X) P_{cd} + P_{ab} P_{cd} g^{bd}(P, X) \beta^e(P, X) \partial_e g^{ac}(P, X) = 0 \tag{2.9}$$

Writing

$$Z_1 = \dot{P}_{ab} \frac{\partial}{\partial p_{ab}} + \dot{X}^a \frac{\partial}{\partial x^a} \qquad \text{and} \qquad Z_2 = P'_{ab} \frac{\partial}{\partial p_{ab}} + X'^a \frac{\partial}{\partial x^a}$$

we find that equation (2.8) takes the form

$$Y \lrcorner Z_1 \lrcorner Z_2 \lrcorner \Omega = 0 \tag{2.10}$$

which is the same as

$$(\dot{P}_{ab} X'^a - P'_{ab} \dot{X}^a) \beta^b + \dot{X}^a X'^b \alpha_{ab} = 0 \tag{2.11}$$

This, using $H = 0$ and equation (2.9), and writing $P^{ab} = g^{ac} g^{bd} P_{cd}$ implies

$$\dot{X}^{[a} X'^{b]} = C P^{ab} \tag{2.12}$$

$$\dot{P}_{ab} X'^a - P'_{ab} \dot{X}^a = C (\partial_b g^{ac}) P_{ad} P_c^{\ d} \tag{2.13}$$

where

$$C = \pm 2T[(\dot{X}X')^2 - \dot{X}^2 X'^2]^{-1/2} \tag{2.14}$$

Note that not all points (p_{ab}, x^c) can lie in the image of dynamically admissible σ's, but only those for which p_{ab} is decomposable, i.e., satisfy the Plücker relation $p_{a[b} p_{cd]} = 0$. It would no doubt be preferable to try to take instead of Γ the "space of true volume elements" in which the Plücker relation is imposed from the outset.

We now demand $s = \pi \circ \sigma$, where π is the canonical projection in $\mathcal{P}$, to be also a diffeomorphism, so that $S = s(Q)$ is again orientable and a consistent choice of sign can be made in (2.14). (In mechanics the analogous assumption is superfluous if spacetime is time-orientable, since in this case $\mathcal{P}$ is known to have two connected components, so that no consistency problem can arise. In the present case $\mathcal{P}$, and also the subset of $\mathcal{P}$ for which p_{ab} is decomposable, is connected.)

Thus, merely from the variational principle (2.6), σ is for some orientation of Q the lift on embedding $s: Q \to M$, and hence $A(\sigma)$ is proportional to the surface area of $S = s(Q)$.

In equation (2.13) we may now write $\dot{P}_{ab} = \dot{X}^c \partial_c P_{ab}$, $P'_{ab} = X'^c \partial_c P_{ab}$, whence

$$2\dot{X}^{[c} X'^{a]} \partial_c P_{ab} = C (\partial_b g^{ac}) P_{ad} P_c^{\ d} \tag{2.15}$$

which, in turn, is the same as

$$P^{ab}\nabla_a P_{bc} = 0 \tag{2.16}$$

Thus, the variational principle (2.6) is equivalent to equation (2.16), supplemented with

$$P^{ab}P_{ab} = -2T^2 \tag{2.17}$$

and the surface S with P_{ab} as volume element is an extremal surface (see Appendix).

Since Θ does not contain dp_{ab} terms, it is clear from the above that one could allow arbitrary variations of p_{ab} on the boundary of Q preserving the constraint (2.3), i.e., only $\pi \circ \sigma$ has to be independent of ε on δQ. If more general variations for σ are allowed on part of the boundary, one obtains the usual open-string boundary conditions on that part. The presence of these boundary conditions slightly complicates the content of the next sections. We shall thus, for simplicity, assume that $Q = S^1 \times \mathbf{R}$ from now on.

3. CONSERVATION LAWS

We first look at transformations which generalize the notion of Hamiltonian vector field in the case of mechanics. A vector field Y on Γ is called canonical iff

$$\mathscr{L}_Y \Omega = 0 \tag{3.1}$$

Since $d\Omega = 0$, this is equivalent to

$$d(Y \lrcorner \Omega) = 0 \tag{3.2}$$

In mechanics, the analogue of (3.2) gives an isomorphism between closed (in that case: 1-) forms of Γ and canonical vector fields Y. In the present context, this is no longer true, since the linear map $\Omega^{\#}: Y \in T\Gamma \to Y \lrcorner \Omega \in \Lambda^2 T^*\Gamma$ is not surjective. Thus, the equation

$$Y \lrcorner \Omega = -dF \tag{3.3}$$

does not have a solution Y for a general 1-form F on Γ. To see what the restrictions are, write

$$Y = \alpha_{ab}(p, x)\frac{\partial}{\partial p_{ab}} + \beta^a(p, x)\frac{\partial}{\partial x^a} \tag{3.4}$$

Then equation (3.2) takes the form

$$\begin{aligned}\alpha_{[ab,c]}\, dx^a \wedge dx^b \wedge dx^c + \alpha_{ab}{}^{,cd}\, dp_{cd} \wedge dx^a \wedge dx^b \\ + 2\beta^a{}_{,c}\, dp_{ab} \wedge dx^c \wedge dx^b - 2\beta^{a,cd}\, dp_{cd} \wedge dp_{ab} \wedge dx^b = 0\end{aligned} \tag{3.5}$$

where we use the convention $\alpha_{,a} = (\partial/\partial x^a)\alpha$ and $\alpha^{,ab} = \alpha^{,[ab]} = (\partial/\partial p_{ab})\alpha$. From the last term in (3.5) we obtain

$$\delta^{[e}{}_f \beta^{a],cd} = \delta^{[c}{}_f \beta^{d],ea} \tag{3.6}$$

After taking suitable contractions, equation (3.6) is seen to imply

$$\beta^{a,bc} = 0 \Rightarrow \beta^a = \beta^a(x) \tag{3.7}$$

Hence, from the first three terms in (3.5), α_{ab} is of the form

$$\alpha_{ab}(p, x) = 2\beta^c{}_{,[a}(x) p_{b]c} + \gamma_{ab}(x) \tag{3.8}$$

with

$$\gamma_{[ab,c]} = 0 \tag{3.9}$$

Thus, Y can only be a linear combination $Y = U + V$ with

$$U = \gamma_{ab}(x) \frac{\partial}{\partial p_{ab}}, \qquad \gamma_{[ab,c]} = 0 \tag{3.10}$$

$$V = 2\beta^c{}_{,a}(x) p_{bc} \frac{\partial}{\partial p_{ab}} + \beta^a(x) \frac{\partial}{\partial x^a} \tag{3.11}$$

U is nothing but the natural shift along the fibers of Γ by the closed 2-form $\gamma_{ab}\, dx^a \wedge dx^b$ on M, whereas V is the canonical lift to Γ of the vector field $\beta^a(x)\, \partial/\partial x^a$ on M. We can now locally solve (3.3) in terms of Γ, obtaining, modulo addition of closed 1-forms on Γ,

$$F_U = -\gamma_a(x)\, dx^a, \qquad \gamma_{[a,b]} = \gamma_{ab} \tag{3.12}$$

$$F_V = -2\beta^a(x) p_{ab}\, dx^a = V \lrcorner \Theta \tag{3.13}$$

Note that F_V exists even globally and satisfies $\mathscr{L}_Y \Theta = 0$ rather than just (3.1).

We now turn to a (continuous) symmetry of the system. This is defined as a canonical vector field Y which is tangent to $\mathscr{P}$. Any such vector field gives rise to a conserved quantity as follows: Let F be the 1-form on Γ generating Y via (3.3). Since the map $\Omega^{\#}$ is injective, Y is unique provided it exists. Let σ be a solution of equation (2.7) and consider the quantity $f = \sigma^* F$ on Q. One finds $df = \sigma^*\, dF = -\sigma^*(Y \lrcorner \omega) = 0$, whence f is closed on Q. Thus, taking a closed cross section $\Sigma \subset Q$, it follows that the quantity

$$m(Y) = \int_\Sigma f \tag{3.14}$$

is in fact independent of Σ. In order to find these quantities explicitly, we have to look at vector fields $Y = U + V$ of the previous section, which satisfy

$$Y(H) = 0 \qquad \text{at points where } H = 0 \tag{3.15}$$

or

$$(\gamma_{ab}+2\beta^c{}_{,a}p_{bc})p^{ab}+p_{ab}p_{cd}g^{ac}{}_{,e}\beta^e g^{bd}=0 \qquad \text{whenever } p_{ab}p^{ab}+2T^2=0 \tag{3.16}$$

Using that γ_{ab}, β^c depend only on x, one easily finds that this can only be true provided γ_{ab} is zero and

$$g^{ab}{}_{,c}\beta^c-2g^{c(b}\beta^{a)}{}_{,c}=0 \tag{3.17}$$

that is to say, $\beta^a\,\partial/\partial x^a$ is a Killing vector of the spacetime M. The associated conserved quantity is now given by equation (3.13). It is instructive to view this quantity in a slightly different manner. Recall that the map σ is obtained as a lift from an embedding s. Use this operation of "lift" to pull back F to a 1-form φ_a on $S=s(Q)$, namely

$$-\varphi_a=2P_{ab}\beta^b \tag{3.18}$$

Conservation along S now takes the form

$$h_{[a}{}^{a'}h_{b]}{}^{b'}\nabla_{a'}\varphi_{b'}=-\frac{1}{2T^2}P_{ab}P^{a'b'}\nabla_{a'}\varphi_{b'}=0 \tag{3.19}$$

where the last equality is easily verified using equation (2.16) and Killing's equation $\nabla_{(a}\beta_{b)}=0$. Note that one has here a direct generalization of the well-known textbook argument concerning conserved quantities along geodesics (Wald, 1984). In the latter case, however, one also finds quantities of quadratic or higher power in momentum when spacetime admits Killing tensors (Sommers, 1973). Unfortunately, these have no analogue here.

4. THE COVARIANT PHASE SPACE $\tilde{\Gamma}$

We now apply the general method of Kijowski and Szczyrba (1976) to derive from the above setting the covariant phase space $\tilde{\Gamma}$ of string theory. We shall be completely formal here and neglect all analytic issues stemming from the infinite dimensionality of $\tilde{\Gamma}$. We first review the formulas in Kijowski and Szczyrba (1976), using, for simplicity, local coordinates in $\mathcal{P}$. Let u^A be such coordinates [where $A=1,\ldots,n+\frac{1}{2}n(n-1)-1=\frac{1}{2}(n^2+n-2)$] and let σ be given locally by $y^\alpha\mapsto\sigma^A(y^\alpha)$. Then the equation $\sigma^*(X\,\lrcorner\,\omega)=0$ takes the form

$$\sigma^A{}_{,\alpha}(y)\sigma^B{}_{,\beta}(y)\omega_{ABC}(\sigma(y))=0 \tag{4.1}$$

Let p be the element of $\tilde{\Gamma}$ corresponding to the map σ. A tangent vector in $\tilde{\Gamma}$ at the point p is geometrically a tangent vector ξ in $\mathcal{P}$ defined along σ in such a way that it connects σ with an infinitesimally nearby solution of equation (4.1). In other words, ξ is given by a solution $\xi^A(y)$ to the linearized version of (4.1), i.e.,

$$2\xi^A{}_{,\alpha}(y)\sigma^B{}_{,\beta}(y)\omega_{ABC}(\sigma(y))+\sigma^A{}_{,\alpha}(y)\sigma^B{}_{,\beta}(y)\omega_{ABC,D}(\sigma(y))\xi^D(y)=0 \tag{4.2}$$

with $\sigma(y)$ corresponding to the given point p in $\tilde{\Gamma}$. Fix a pair of solutions (ξ^A, η^A) to equation (4.2) and consider the 1-form on Q given by $\sigma^*(\xi \lrcorner \eta \lrcorner \omega)$. Now define the following integral W over some closed cross section Σ of $Q \cong S^1 \times \mathbf{R}$:

$$W(\xi, \eta)=\int_\Sigma \sigma^*(\xi \lrcorner \eta \lrcorner \omega) \tag{4.3}$$

Using equations (4.1) and (4.2) and the fact that $d\omega$ is zero, it is now a straightforward matter to prove that $d(\sigma^*(\xi \lrcorner \eta \lrcorner \omega))$ is zero on Q. Thus, $W(\xi, \eta)$ in (4.3) is independent of Σ and hence a well-defined 2-form on $\tilde{\Gamma}$. It is equally simple to show that the 3-form on $\tilde{\Gamma}$ given by the formal exterior derivative of W vanishes. Thus, $(\tilde{\Gamma}, W)$ is a presymplectic space. One expects W to be degenerate with respect to vectors at the point $p \in \tilde{\Gamma}$ corresponding to a reparametrization of σ, viewed as a surface in $\mathscr{P}$. But this easily follows from equation (4.3), since such vectors at p are just vector fields ξ along σ which are tangent to σ, i.e., $\xi^A = f^A{}_{,\alpha}c^\alpha$, and for those

$$\begin{aligned}
\tfrac{1}{6}\sigma^*(\xi \lrcorner \eta \lrcorner \omega) &= \sigma^C{}_{,\alpha}\xi^A(y)\eta^B(y)\omega_{ABC}(\sigma(y))\,dy^\alpha \\
&= \sigma^C{}_{,\alpha}(y)\sigma^A{}_{,\beta}(y)c^\beta(y)\omega_{ABC}(\sigma(y))\eta^B(y)\,dy^\alpha \\
&= c^\beta(y)\cdot 0 = 0
\end{aligned} \tag{4.4}$$

We now proceed to explicitly evaluate W in the present circumstances, where $u^A=\sigma^A(y^\alpha)$ corresponds to $(p_{ab}=P_{ab}(y),\ x^c=X^c(y))$ satisfying

$$P^{ab}=2T[(\dot{X}X')^2-\dot{X}^2X'^2]^{-1/2}\dot{X}^{[a}X'^{b]} \tag{4.5}$$

and the equation containing $\dot{P}_{ab}$ and P'_{ab} is not needed at the moment. In order to obtain $\xi^A(y)=\delta\sigma^A(y)=(\delta P_{ab}(y), \delta X^c(y))$, we have to vary equation (4.5), thus obtaining a relation between δP_{ab} and δX^c. For simplicity, we write this down only in the case where g_{ab} is flat. It is clear that the general case is obtained by replacing partial derivatives by covariant ones in the final expression for W. We obtain

$$\delta P_{ab} = -2P^c{}_{[a}U_{b]d}\delta X^d{}_{,c} \tag{4.6}$$

where $U^a{}_b=\delta^a{}_b-h^a{}_b$ is the operator projecting on the spacelike $(n-2)$-space orthogonal to the world sheet S in M.

We now have to evaluate the integrand in equation (4.3), using

$$\xi^A=(\delta P_{ab}, \delta X^c=\xi^c), \qquad \eta^A=(\delta P'_{ab}, \delta X'^c=\eta^c)$$

subject to equation (4.6). Since ξ^A and η^A of this form are already tangential to $\mathscr{P}$, we have

$$\sigma^*(\xi \lrcorner \eta \lrcorner \omega) = \sigma^*(\xi \lrcorner \eta \lrcorner \Omega) \tag{4.7}$$

Using equation (2.2), we find

$$\xi \lrcorner \eta \lrcorner \Omega = 2\xi^a \delta P'_{ab}\, dx^b - 2\eta^a \delta P_{ab}\, dx^b + 2\xi^a \eta^b\, dp_{ab} \tag{4.8}$$

Note that (4.8) is to be viewed as a 1-form on $\mathscr{P}$, defined on the image of σ. We now insert (4.6) into (4.8) and pull back the resulting expression by the map σ. In the spirit of the Appendix, we write the resultant 1-form on Q as a 1-form w_a on M which is purely tangential to $S = s(Q)$. With this understanding we get

$$-\tfrac{1}{2} w_a(\xi, \eta) = (P_a{}^b U^{cd} - h_a{}^b P^{cd})(\xi_c \nabla_b \eta_d - \eta_c \nabla_b \xi_d) - h_a{}^b \nabla_b (P_{cd} \xi^c \eta^d) \tag{4.9}$$

where we have reintroduced the covariant derivative ∇_a in order to cover the general case where g_{ab} can be curved. The last term in (4.9) does not contribute to $W(\xi, \eta)$. Using the methods of the Appendix, it is now instructive to verify explicitly what in fact we know from general principles, namely that the integral $\int_\Sigma w_a(\xi, \eta)\, dx^a$ over a closed cross section Σ of S vanishes when ξ or η is tangent to S. We leave this as an exercise. On the other hand, checking that $\int_\Sigma w_a\, dx^a$ is independent of Σ, that is to say,

$$P^{ab} \nabla_a w_b = 0 \tag{4.10}$$

is not so easy, since this involves the complicated differential equation satisfied by ξ^a and η^a which results from the linearization of equation (2.13), or, equivalently, $P^{ab} \nabla_a P_{bc} = 0$. This of course is just the Jacobi equation for minimal surfaces which can be found in the literature (Simons, 1968). An easy derivation along the lines of the standard textbook argument for geodesics (Wald, 1984) could run as follows:

Since the part of ξ which is tangential to S does not contribute to W, we can assume ξ to be orthogonal to S, i.e., $U^a{}_b \xi^b = \xi^b$. We can in addition choose it in such a way that

$$\mathscr{L}_\xi P^{ab} = 0 \tag{4.11}$$

(Of course, as opposed to previous sections, $\mathscr{L}_\xi$ is now the Lie derivative in M.) Using (A.1)-(A.3) (4.11), and commuting derivatives, it is now straightforward though tedious to show that

$$U^a{}_f p^d{}_b \nabla_d (p^{bc} U^f{}_h \nabla_c \xi^h) = K^a_{bd} h^{bc} h^{de} K^f_{ce} \xi_f - U^a{}_b R^b{}_{cde} h^{cd} \xi^e \tag{4.12}$$

where K^a_{bc} is the extrinsic curvature of S and $R^a{}_{bcd}$ is the curvature tensor defined by $2\nabla_{[a} \nabla_{b]} \alpha_c = R_{abc}{}^d \alpha_d$. The left side in equation (4.12) is essentially the d'Alembertian on S acting on vector fields normal to S. Since S is timelike, this is a linear hyperbolic equation for ξ^a.

When both ξ and η are orthogonal to S, equation (4.9), modulo an exact form, becomes

$$-\tfrac{1}{2}w_a(\xi, \eta) = P_a{}^b(\xi^c \nabla_b \eta_c - \eta^c \nabla_b \xi_c) \tag{4.13}$$

Using (4.12) and the symmetry properties of K^a_{bc} and $R_{abc}{}^d$, one now verifies equation (4.10) easily.

It is clear that the analogue of equation (4.13) for m-dimensional objects $(m < n)$ just involves adding $m-1$ indices to P_{ab} for the volume element of the m-worldsurface S swept out by the m-brane. This yields the m-form $w_{a_1 \cdots a_m}$ whose integral over m-dimensional cross sections of S is the presymplectic form on the covariant phase space of m-branes.

This ends our derivation of the covariant phase space of string theory. One might wonder how it comes that one has apparently just one constraint in the present canonical theory, namely $P_{ab}P^{ab} = -2T^2$, while there are two constraints in the standard one (Scherk, 1975). The answer is that the variables of the standard phase space are obtained by decomposing the covariant ones in such a way that the equation $P^{ab}\nabla_a P_{bc} = 0$ is turned into an initial-value problem. It is in this process that the remaining constraint emerges.

5. CONCLUSION AND COMMENTS

In this paper we have rephrased the theory of the classical bosonic string in arbitrary spacetimes in the setting of a particular variant of multisymplectic geometry. We have argued that this scheme has the advantage of making manifest the relevant structures of the theory and putting greater emphasis on the analogy with the relativistic particle than the standard formulation. We have seen that the present approach lends itself easily to a systematic search for conserved quantities. It was gratifying to see that an elegant version of the infinite-dimensional space of strings can be found directly from the multisymplectic framework.

A number of fairly obvious generalizations suggest themselves: One can study m-branes by considering $\Lambda^{m+1}T^*M$ instead of $\Lambda^2 T^*M$, and one could introduce interactions by introducing potentials into the Hamiltonian H. One could perhaps dispose of the spacetime metric g_{ab} by showing that, e.g., for strings, one really needs only the "areal" metric on the antisymmetric 2-tensors derived from it, namely $g_{c[a}g_{b]d}$.

While the present formulation is useful in such questions regarding the general structure of the theory, it would be much more interesting to refine this formulation in such a way as to shed light on some of the more specific properties of strings, such as the ("complete") integrability of the equations when M is flat or, more ambituiously, finding a "finite-dimensional" (in the phase space sense) origin for the occurrence of critical spacetime dimensions in the quantum theory.

As for the multisymplectic formalism in general, one should perhaps keep in mind that, despite its elegance, the replacement of the canonical 2-form of mechanics by a canonical $(m+1)$-form (for a field theory with m independent variables) is a technically conservative step, but, largely due to the resultant asymmetry between coordinate-type and momentum-type quantities, conceptually a radical step which should consequently be judged by its potential to make radical predictions or treating problems which are intractable by more conventional means. Most importantly, one would like to see a new kind of quantum theory emerging from the multisymplectic formalism, which is not beset with the same problems as standard quantum field theory. Whether this is possible is, however, completely unclear at present.

APPENDIX. TIMELIKE 2-SURFACES

Let Q be an orientable 2-manifold with local coordinates y^α $(\alpha=1,2)$. Take a timelike 2-surface $S\subset M$ given as the image of an embedding s of Q into M. Locally, s can be written as $y^\alpha \mapsto x^a = X^a(y^\alpha)$. Define $\dot{X}^a = (\partial/\partial y^1)X^a$, $X'^a = (\partial/\partial y^1)X^a$. Taking an orientation for Q and using some normalization, one has a fixed surface element P^{ab} on S, which is given as a unique nonzero function times $\dot{X}^{[a}X'^{b]}$. It is computationally convenient to imagine P^{ab} being given as a tensor field defined on a whole neighborhood of S in M. Of course, all formulas depend only on the values of P^{ab} at—and its derivatives along—the surface S. We are thus led to look at tensors $P^{ab}=P^{[ab]}$ which satisfy

$$P^{a[b}P^{cd]}=0 \tag{A.1}$$

$$P^{d[a}\nabla_d P^{bc]}=0 \tag{A.2}$$

$$P^{ab}P_{ab}=-2T^2 \tag{A.3}$$

(A.1) states the decomposability of P^{ab} into the exterior product of two vectors which, according to (A.2), are surface-forming. [Note that ∇_a in (A.2) could be any torsion-free derivative operator.] Equation (A.3) is a normalization convention consistent with string terminology where T plays the role of string tension, the sign being determined by the timelike character of S. The intrinsic metric h_{ab} of the surface spanned by P^{ab} is given by

$$h_{ab}=\frac{1}{T^2}P_{ac}P^c{}_b \tag{A.4}$$

The mixed object $h_a{}^b$ plays the role of the operator projecting on directions tangential to S. Let λ^a be tangential to S, i.e.,

$$\lambda^a = h^a{}_b \lambda^b \tag{A.5}$$

Then the quantity $h_a{}^{a'}(\nabla_{a'}\lambda^b - h_{b'}{}^b \nabla_{a'}\lambda^{b'})$ is "tensorial with respect to λ^a," i.e., independent of $\nabla_a \alpha$ when λ^α is replaced by $\alpha\lambda^a$. Thus, there exists a tensor K^c_{ab} such that

$$h_a{}^{a'}\nabla_{a'}\lambda^b = h_a{}^{a'} h_{b'}{}^b \nabla_{a'}\lambda^{b'} + K^b_{ac}\lambda^c \tag{A.6}$$

Obviously $h_{ab}K^b_{ac} = 0$. Inserting (A.5) into the left side of (A.6), we find

$$\begin{aligned} T^2 K^a_{bc} &= T^2 h_b{}^{b'} h_c{}^{c'} \nabla_{b'} h_{c'}{}^a \\ &= h_b{}^{b'} h_c{}^{c'} (P_{c'd}\nabla_{b'} P^{da} + P^{da}\nabla_{b'} P_{c'd}) \\ &= h_b{}^{b'} P_{cd} \nabla_{b'} P^{da} - 2 h_b{}^{b'} h_c{}^a P^{c'd} \nabla_{b'} P_{c'd} \\ &= h_b{}^{b'} P_{cd} \nabla_{b'} P^{da} \end{aligned} \tag{A.7}$$

where we have used (A.1) in the third line and (A.3) in the fourth line. Next, contract (A.7) with P^{bc}. One obtains

$$T^2 P^{bc} K^a_{bc} = P^d{}_c P^{b'c} \nabla_{b'} P_d{}^a = -P^d{}_c P^{b'}{}_d \nabla_{b'} P^{ac} = -T^2 P^{bc} K^a_{bc} = 0 \tag{A.8}$$

where (A.2) and (A.3) have been used in the third line. Thus, it follows that $K^a_{bc} = K^a_{(bc)}$. K^a_{bc} is called the second fundamental tensor or extrinsic curvature. S is an extremal surface iff $h^{ab}K^c_{ab} = 0$ or, equivalently,

$$P^{ab}\nabla_a P_{bc} = 0 \tag{A.9}$$

The extension of these considerations to surfaces of arbitrary dimension (and hence membranes, etc.) is fairly obvious.

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Note added. After completion of this work I learnt that the idea of using $\Gamma = \Lambda^{m+1} T^* M$ as the multi-phase space of m-dimensional extended objects appears already in W. M. Tulczyjew, *Annales de l'Institut Henri Poincaré*, **34A**, 25 (1981). However, the applications in Sections 3 and 4 are new.

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